Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Summer Examination-2018

Subject Name : Ring Theory

Subject Code : 4SC06RTC1 **Branch: B.Sc. (Mathematics)**

Semester : 6 **Date : 27/04/2018** **Time : 02:30 To 05:30** **Marks : 70**

Instructions:

(1) Use of Programmable calculator & any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

Q-1 **Attempt the following questions:** **(14)**

a) Define: Division ring

b) Find zero divisors of the ring $(Z_7;\ +_7;\ \cdot_7)$.

c) Find characteristic of the ring $(Z_6;\ +_6;\ \cdot_6)$.

d) Let I be an ideal in ring R with unity. If $a \in R$ is unit in R then show that $a \in I$ implies $I = R$.

e) Define: Quotient ring

f) For the ideal $I = 7Z$ in the ring $(Z;\ +;\ \cdot)$, if A is any ideal of Z with $I \subset A$, then prove that either $A = I$ or $A = Z$.

g) Show that the ring $(P(U);\ \Delta;\ \cap)$ is not an integral domain if U contains more than one element.

h) Define: Field

i) Prove that the mapping $\phi:\left(Z\left(\sqrt{2}\right);\ +;\ \cdot\right) \to \left(Z\left(\sqrt{2}\right);\ +;\ \cdot\right)$ where $\phi(a) = m - n\sqrt{2}$ for $a = m + n\sqrt{2} \in Z\left(\sqrt{2}\right)$ is a homomorphism.

j) If the homomorphism $\phi:(Z;\ +;\ \cdot) \to (Z_2;\ +_2;\ \cdot_2)$ is defined as

$$\phi(x) = \begin{cases} 0 & ;x \text{ is even} \\ 1 & ;x \text{ is odd} \end{cases}$$

then find kernel of ϕ.

k) Show that an integral domain contains no idempotent except 0 or 1.

l) Show that the polynomial $x^2 + 1$ is irreducible as an element of $Q[x]$ but reducible as an element of $Z_5[x]$.

m) Find the g.c.d of $f(x) = x^3 + 3x^2 + 3x + 3$ and $g(x) = 4x^3 + 2x^2 + 2x + 2 \in Z_5[x]$ and express it in the form $a(x)f(x) + b(x)g(x)$.

n) Find zeros of $f(x) = 3x^4 + 5x^2 + x$ in $Z_5[x]$.

Attempt any four questions from Q-2 to Q-8

Q-2		**Attempt all questions**	**(14)**
	a)	State and prove properties of Ring.	**(5)**
	b)	Prove that the restricted cancellation law for multiplication holds good in a commutative ring iff it has no zero divisors.	**(5)**
	c)	Show that for ideals I_1 and I_2 of a ring R, $I_1 \cup I_2$ is also an ideal of R iff either $I_1 \subset I_2$ or $I_2 \subset I_1$.	**(4)**
Q-3		**Attempt all questions**	**(14)**
	a)	If $I = 4Z$ in the ring $R = (Z;\ +;\ \cdot)$ then prepare addition and multiplication table for quotient ring R/I.	**(5)**
	b)	Let $(M_2(Z);\ +;\ \cdot)$ be a ring. Check whether the $I = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| a, b \in Z \right\}$ is an ideal of $M_2(Z)$ or not.	**(5)**
	c)	Prove that a field is an integral domain.	**(4)**
Q-4		**Attempt all questions**	**(14)**
	a)	For a given subrings U_1 and U_2 of a ring R, Show that their intersection $U_1 \cap U_2$ is also a subring of R.	**(5)**
	b)	If a commutative ring *R* with unity has no proper ideal, then prove that *R* is a field.	**(5)**
	c)	Obtain the quotient $q(x)$ and remainder $r(x)$ on dividing $f(x) = 4x^4 - 3x^2 + 2$ by $g(x) = x^3 - 2x + 1$ in $R[x]$ and express $f(x)$ in the form $q(x)g(x) + r(x)$.	**(4)**
Q-5		**Attempt all questions**	**(14)**
	a)	Find the g.c.d. of $f(x) = 6x^3 + 5x^2 - 2x + 25$ and $g(x) = 2x^2 - 3x + 5 \in R[x]$ and express it in the form $a(x)f(x) + b(x)g(x)$.	**(5)**
	b)	Prove that for nonzero polynomials, $f, g \in D[x]$, $[fg] = [f] + [g]$.	**(5)**
	c)	Prove that a homomorphism defined on the ring $(Z;\ +;\ \cdot)$ is either a zero homomorphism or identity mapping.	**(4)**
Q-6		**Attempt all questions**	**(14)**
	a)	If for $f(x) = (1, -2, 0, 3, 0, \ldots\ldots\ldots)$ and $g(x) = (2, 0, -3, 0, 4, 0, \ldots\ldots\ldots) \in Z[x]$ then find $f(x) + g(x)$ and $f(x) \cdot g(x)$.	**(5)**
	b)	Obtain all principal ideals in the ring $(Z_{12};\ +_{12};\ \cdot_{12})$.	**(5)**
	c)	Let *a* and *b* be two elements of an integral domain *D*. If for relatively prime integers *m* and *n*, $a^m = b^m$ and $a^n = b^n$ then show that $a = b$.	**(4)**
Q-7		**Attempt all questions**	**(14)**
	a)	If we define addition and multiplication in power set $P(U)$, U being the universal set, as follows: For $A, B \in P(U)$ $A + B = A \Delta B = (A \cup B) - (A \cap B)$ $A \cdot B = A \cap B$ then show that $(P(U);\ +;\ \cdot)$ is a ring.	**(7)**

b) Show that the characteristic of a ring R with unity is n iff n is the smallest positive integer with $n1 = 0$. **(7)**

Q-8 **Attempt all questions** **(14)**

a) Show that a nonempty subset U of a ring R is a subring of R iff the following conditions are satisfied. **(7)**

(i) $a - b \in U$ and (ii) $ab \in U$ for $a, b \in U$

b) If $(R;\ +;\ \cdot)$ is a ring wih unity then show that the mapping **(7)**

$\phi : (Z;\ +;\ \cdot) \to (R;\ +;\ \cdot)$, where $\phi(n) = n1,\ n \in Z$, is a homomorphism with

(i) $K_\phi = \langle m \rangle$ if the characteristic of R is m, and

(ii) $K_\phi = \{0\}$ if the characteristic of R is zero.